

Superconformal Algebras in Light-cone Gauge Quantization of String Theories on AdS_3

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Abstract

Motivated by superstring theories on AdS_3 , we construct spacetime superconformal algebras (SCAs) living on the AdS_3 boundary in terms of the transversal physical degrees of freedom. The SCAs constructed are N=4 large, middle algebras, and $N = 3$ algebra, corresponding to superstring theories on $AdS_3 \times S^3 \times S^3 \times S^1$, $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times (S^3 \times S^3 \times S^1)/Z_2$ backgrounds respectively.

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1 Introduction

String theory on AdS space has been studied intensively, [1, 2, 3, 4] in particular, with the connection to Maldacena's conjecture [5, 11, 12, 13, 14, 6, 7, 8, 9, 10]. A precise statement on the AdS/CFT correspondence has been given in ref. [15, 16]. In ref.[17, 18], the spacetime CFT living on the boundary of AdS_3 were studied and the $N = 4$ superconformal algebras on the spacetime boundary were constructed. Spacetime properties of superstring theory on $AdS_3 \times S^3 \times S^3 \times S^1$ were studied in ref.[20], and the "large" $N = 4$ algebras were constructed. Both of those algebras were constructed in the covariant form. In ref.[21] the light-cone gauge quantization of string theory on AdS_3 was given and shown to be equal to the covariant one, provided the center charge is 26 (Bosonic) or 15 (Fermionic). In the light-cone gauge, the spacetime superconformal algebra (SCA) were constructed in terms of the transversal physical degrees of freedom [21]. In this note we shall go along this line further. That is, we shall construct spacetime large, middle $N = 4$ [22, 23] and $N = 3$ [24] SCAs in terms of the transversal physical degrees of freedom, corresponding to superstring theories on $AdS_3 \times S^3 \times S^3 \times S^1$, $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times (S^3 \times S^3 \times S^1)/Z_2$ spacetime background respectively. This note is organized as follows. In section 2, we introduce our notations first, then we construct the large $N = 4$ SCF algebra. In section 3, we get middle $N = 4$ SCF algebra as well as $N = 3$ SCF algebra from the large one by taking certain limits. Some remarks are given in the last section.

2 Large $N = 4$ spacetime SCFT

In this section we construct large $N = 4$ SCF algebra, which corresponds to string propagating on $AdS_3 \times S^3 \times S^3 \times S^1$ background. In Neveu-Schwarz-Ramond formalism, the worldsheet supersymmetry introduces ten worldsheet Majorana fermions, which however, transform as spacetime vectors. There are three worldsheet fermions ψ^A , "A" being the vector indices on the $SL(2)$ manifold, and similarly three fermions χ^a from the first $SU(2)$ manifold, three fermions ω^a from the second $SU(2)$ manifold, and a single fermion λ from the $U(1)$ part. Our convention are almost the same as in ref. [20] and [21]. The OPEs among them are:

$$\begin{aligned} \psi^A(z)\psi^B(w) &= \frac{\eta^{AB}}{z-w} \\ \chi^a(z)\chi^b(w) &= \omega^a(z)\omega^b(w) = \frac{\delta^{ab}}{z-w} \\ \lambda(z)\lambda(w) &= \frac{1}{z-w} \end{aligned} \tag{1}$$

where $\eta^{AB} = \text{diag}(+, +, -)$.

In the $SL(2) \times SU(2) \times SU(2) \times U(1)$ super WZNW model, there are affine bosonic currents, j^A, k^i, m^i with level $k+2, k'-2, k''-2$, respectively. The OPEs among them are

$$\begin{aligned} j^A(z)j^B(w) &= \frac{\frac{k+2}{2}\eta^{AB}}{(z-w)^2} + \frac{i\eta_{CD}\epsilon^{ABC}j^D}{z-w} \\ k^i(z)k^j(w) &= \frac{\frac{k'-2}{2}\delta^{ij}}{(z-w)^2} + \frac{i\epsilon^{ijk}k^k}{z-w} \end{aligned}$$

$$\begin{aligned}
m^i(z)m^j(w) &= \frac{\frac{k''-2}{2}\delta^{ij}}{(z-w)^2} + \frac{i\epsilon^{ijk}m^k}{z-w} \\
\partial Y(z)\partial Y(w) &= -\frac{1}{(z-w)^2}
\end{aligned} \tag{2}$$

The criticality of fermionic string, $c = 15$, gives the following relation [20]

$$\frac{1}{k} = \frac{1}{k'} + \frac{1}{k''} \tag{3}$$

The $sl(2)$ current algebra can be constructed in terms of the three bosons as follows,

$$\begin{aligned}
j^3 &= \beta\gamma + \frac{\alpha_+}{2}\partial\phi \\
j^+ &= \beta\gamma^2 + \alpha_+\gamma\partial\phi + k\partial\gamma \\
j^- &= \beta
\end{aligned} \tag{4}$$

where $\alpha_+ = \sqrt{2k}$. The OPEs of β, γ, ϕ are

$$\begin{aligned}
\phi(z)\phi(w) &= -\log(z-w) + \dots\dots\dots \\
\beta(z)\gamma(w) &= \frac{1}{z-w} + \dots\dots\dots
\end{aligned} \tag{5}$$

Here, $j^+ = j^1 + ij^2$, $j^- = j^1 - ij^2$.

When the light-cone gauge [21] $\gamma = z^p$, and $\tilde{\psi}^+ = 0$ is imposed, only ϕ and $\tilde{\psi}^3$ (The OPE can be normalized as $\tilde{\psi}^3(z)\tilde{\psi}^3(w) = \frac{1}{z-w}$) from AdS_3 part are dynamical ones. The bosonic part of the spacetime Virasoro generators are

$$L_n = \oint dz \left[-\frac{1}{2}\partial\phi\partial\phi - \left(\frac{1}{\alpha_+} - \frac{\alpha_+}{2}\right)\partial^2\phi + \frac{1}{k'}\sum k^i k^i + \frac{1}{k''}\sum m^i m^i - \frac{1}{2}\partial Y\partial Y - \frac{\Delta}{z^2} \right] e^{nq} z^{np+1} \tag{6}$$

Now we decompose the transversal physical degrees of freedom into the representations of the $su(2) \times su(2)$. The Liouville field ϕ and its super partner ($\tilde{\psi}^3$) from the AdS_3 are in the $(\mathbf{0}, \mathbf{0})$. The k^i 's and their super partners χ^a 's are in the $(\mathbf{1}, \mathbf{0})$. Fields from $U(1)$ and its super partner are in the $(\mathbf{0}, \mathbf{0})$, and m^i 's and their super partners ω^a in the $(\mathbf{0}, \mathbf{1})$. Since the spacetime supercurrents $G_{\alpha\beta}$'s are spacetime spinors, and the worldsheet fermions are spacetime vectors, we first need to transform the later into spacetime spinors. This can be done by the usual bosonization procedure, as in ref. [21]

Define

$$\begin{aligned}
\chi^3 + \tilde{\psi}^3 &= e^{\phi^1} \\
\chi^3 - \tilde{\psi}^3 &= e^{-\phi^1} \\
\chi^+ &= e^{\phi^2} \\
\chi^- &= e^{-\phi^2} \\
\lambda^3 + i\lambda^0 &= e^{\phi^3} \\
\lambda^3 - i\lambda^0 &= e^{-\phi^3} \\
\lambda^1 + i\lambda^2 &= e^{\phi^4} \\
\lambda^1 - i\lambda^2 &= e^{-\phi^4}
\end{aligned} \tag{7}$$

Then we can construct the global $su(2) \times su(2)$ generators acting on these fermions in terms of the ϕ^A 's

$$\begin{aligned}
j^+ &= \oint e^{\phi^2} (e^{\phi^1} + e^{-\phi^1}) \\
j^- &= \oint (e^{\phi^1} + e^{-\phi^1}) e^{-\phi^2} \\
j^3 &= \oint \partial \phi^2 \\
k^+ &= \oint e^{\phi^4} (e^{\phi^3} + e^{-\phi^3}) \\
k^- &= \oint (e^{\phi^3} + e^{-\phi^3}) e^{-\phi^4} \\
k^3 &= \oint \partial \phi^4
\end{aligned} \tag{8}$$

Using the ϕ^A fields and considering the representations of the j^a 's and k^a 's, we can construct a set of worldsheet fermions which are also spacetime spinors

$$\begin{aligned}
\psi_1 &= e^{\frac{1}{2}(\phi_1 + \phi_2 + \phi_3 + \phi_4)} \\
\bar{\psi}_1 &= e^{\frac{1}{2}(-\phi_1 - \phi_2 - \phi_3 - \phi_4)} \\
\psi_2 &= e^{\frac{1}{2}(-\phi_1 - \phi_2 + \phi_3 + \phi_4)} \\
\bar{\psi}_2 &= e^{\frac{1}{2}(\phi_1 + \phi_2 - \phi_3 - \phi_4)} \\
\chi_1 &= e^{\frac{1}{2}(-\phi_1 + \phi_2 + \phi_3 - \phi_4)} \\
\bar{\chi}_1 &= e^{\frac{1}{2}(\phi_1 - \phi_2 - \phi_3 + \phi_4)} \\
\chi_2 &= e^{\frac{1}{2}(\phi_1 - \phi_2 + \phi_3 - \phi_4)} \\
\bar{\chi}_2 &= e^{\frac{1}{2}(-\phi_1 + \phi_2 - \phi_3 + \phi_4)}
\end{aligned} \tag{9}$$

where ϕ_i satisfy $\phi_i(z)\phi_j(w) = \delta_{ij} \log(z-w)$, and the OPEs of ψ_α , χ_α are

$$\psi_\alpha(z)\bar{\psi}_\beta(w) = \frac{\delta_{\alpha\beta}}{z-w} + \dots \tag{10}$$

$$\chi_\alpha(z)\bar{\chi}_\beta(w) = \frac{\delta_{\alpha\beta}}{z-w} + \dots \tag{11}$$

and all others are regular. It is easy to see that both ψ_i 's and χ_i 's are doublets of j^i and k^i , j^i acts from the left, and k^i from the right [21]. The contributions to the spacetime energy-momentum tensor from fermions in terms of ψ 's, $\bar{\psi}$'s and χ 's, $\bar{\chi}$'s can be expressed as [21]

$$T_F = \frac{1}{2}(\partial\psi_1\bar{\psi}_1 - \psi_1\partial\bar{\psi}_1 + \partial\psi_2\bar{\psi}_2 - \psi_2\partial\bar{\psi}_2 + \partial\chi_1\bar{\chi}_1 - \chi_1\partial\bar{\chi}_1 + \partial\chi_2\bar{\chi}_2 - \chi_2\partial\bar{\chi}_2) \tag{12}$$

where we have set $q=0, p=1$, so we can regard

$$T = -\frac{1}{2}\partial\phi\partial\phi - \left(\frac{1}{\alpha_+} - \frac{\alpha_+}{2}\right)\partial^2\phi + \frac{1}{k'}\sum k^i k^i + \frac{1}{k''}\sum m^i m^i - \frac{1}{2}\partial Y \partial Y + T_F$$

as the energy-momentum tensor in spacetime.

Now, we have eight bosons and eight fermions. But this set of variables are not convenient.

For future convenience, we introduce two new variables to substitute for $\partial\phi$ and ∂Y . Define

$$J^0 = A\partial\phi + B\partial Y, \quad K^0 = C\partial\phi + D\partial Y \quad (13)$$

and

$$J^0(z)J^0(w) = K^0(z)K^0(w) = -\frac{1}{(z-w)^2}, \quad J^0(z)K^0(w) = 0 \quad (14)$$

The coefficients A, B, C, D satisfy

$$A^2 + B^2 = C^2 + D^2 = AD - BC = 1, \quad AC + BD = 0 \quad (15)$$

we can choose the solution

$$A = D = \sqrt{\gamma}, \quad B = -C = -\sqrt{1-\gamma} \quad (16)$$

where $\gamma = \frac{k''}{k'+k''}$. Of course both J^0 and K^0 belong to $(\mathbf{0}, \mathbf{0})$. So the energy-momentum tensor in spacetime can be rewritten as

$$T(z) = -\frac{1}{2}(J^0)^2 - \frac{1}{2}(K^0)^2 - \left(\frac{1}{\alpha_+} - \frac{\alpha_+}{2}\right)\sqrt{\gamma}\partial J^0 - \left(\frac{1}{\alpha_+} - \frac{\alpha_+}{2}\right)\sqrt{1-\gamma}\partial K^0 + \frac{1}{k'}\sum k^i k^i + \frac{1}{k''}\sum m^i m^i + T_F \quad (17)$$

Now we are ready to construct the large $N = 4$ algebra, which consists of sixteen holomorphic currents [22, 20]. Apart from the energy-momentum tensor $T(z)$ and its four superpartners $G_{\alpha\beta}$, there are seven currents A^i, B^i, U , which generate $su(2) \times su(2) \times u(1)$ subalgebra, (it arises from the $S^3 \times S^3 \times S^1$ background, and is the so-called "R-symmetry" of the large $N = 4$ algebra. The affine $su(2) \times su(2) \times u(1)$ algebra in the spacetime $N = 4$ SCFT is lifted from the worldsheet.). Finally, there are four weight $\frac{1}{2}$ fermionic generators $Q_{\alpha\beta}$. They satisfy the following OPEs

$$\begin{aligned} T(z)T(w) &= \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \\ G_{\alpha\beta}(z)G_{\alpha'\beta'}(w) &= \epsilon_{\alpha\alpha'}\epsilon_{\beta\beta'}\left[\frac{2c/3}{(z-w)^3} + \frac{2T(w)}{z-w}\right] + (\sigma_i)_\alpha{}^\rho\epsilon_{\rho\alpha'}\epsilon_{\beta\beta'}\left[\frac{4\gamma A^i}{(z-w)^2} + \frac{2\gamma\partial A^i}{z-w}\right](w) \\ &\quad + (\sigma_i)_{\beta'}{}^\rho\epsilon_{\rho\beta}\epsilon_{\alpha\alpha'}\left[\frac{4(1-\gamma)B^i}{(z-w)^2} + \frac{2(1-\gamma)\partial B^i}{z-w}\right](w) \\ A^i(z)G_{\alpha\beta}(w) &= \frac{1}{2}(\sigma_i)_\alpha{}^\rho\left[\frac{G_{\rho\beta}}{z-w} - \frac{2(1-\gamma)Q_{\rho\beta}}{(z-w)^2}\right](w) \\ B^i(z)G_{\alpha\beta}(w) &= \frac{1}{2}(\sigma_i)_\beta{}^\rho\left[\frac{G_{\alpha\rho}(w)}{z-w} + \frac{2\gamma Q_{\alpha\rho}(w)}{(z-w)^2}\right] \\ A^i(z)A^j(w) &= \frac{k'/2\delta^{ij}}{(z-w)^2} + \frac{i\epsilon^{ijk}A^k(w)}{z-w} \\ B^i(z)B^j(w) &= \frac{k''/2\delta^{ij}}{(z-w)^2} + \frac{i\epsilon^{ijk}B^k(w)}{z-w} \\ Q_{\alpha\beta}(z)G_{\alpha'\beta'}(w) &= \frac{\epsilon_{\alpha\alpha'}\epsilon_{\beta\beta'}U(w)}{z-w} + \frac{1}{z-w}\left[(\sigma_i)_\alpha{}^\rho\epsilon_{\rho\alpha'}\epsilon_{\beta\beta'}A^i - (\sigma_i)_{\beta'}{}^\rho\epsilon_{\rho\beta}\epsilon_{\alpha\alpha'}B^i\right](w) \end{aligned}$$

$$\begin{aligned}
A^i(z)Q_{\alpha\beta}(w) &= \frac{1}{2}(\sigma_i)_\alpha{}^\rho \frac{Q_{\rho\beta}(w)}{z-w} \\
B^i(z)Q_{\alpha\beta}(w) &= \frac{1}{2}(\sigma_i)_\beta{}^\rho \frac{Q_{\alpha\rho}(w)}{z-w} \\
U(z)G_{\alpha\beta}(w) &= \frac{Q_{\alpha\beta}(w)}{(z-w)^2} \\
U(z)U(w) &= -\frac{c}{12\gamma(1-\gamma)(z-w)^2} \\
Q_{\alpha\beta}(z)Q_{\alpha'\beta'}(w) &= -\frac{c}{12\gamma(1-\gamma)} \frac{\epsilon_{\alpha\alpha'}\epsilon_{\beta\beta'}}{z-w} \\
T(z)\Phi(w) &= \frac{d_\Phi\Phi(w)}{(z-w)^2} + \frac{\partial\Phi(w)}{z-w}
\end{aligned} \tag{18}$$

where $\sigma_i = (\sigma^i)^*$, σ^i , $i = 1, 2, 3$ is the Pauli matrix,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{19}$$

$\Phi(z) = \{G_{\alpha\beta}, A^i, B^i, U, Q_{\alpha\beta}\}$ and $d_\Phi = \{\frac{3}{2}, 1, 1, 1, \frac{1}{2}\}$ accordingly, $c = 6k$.

We begin with the construction of the $su(2) \times su(2) \times u(1)$ subalgebra. As in [21, 19], the worldsheet current algebra pertaining to the first $su(2)$ can be represented as

$$A^i = k^i + \frac{1}{2}\psi_\alpha\sigma_{\alpha\beta}^i\bar{\psi}_\beta + \frac{1}{2}\chi_\alpha\sigma_{\alpha\beta}^i\bar{\chi}_\beta \tag{20}$$

the explicit expression is

$$\begin{aligned}
A^+ &= k^+ + \psi_1\bar{\psi}_2 + \chi_1\bar{\chi}_2 \\
A^3 &= k^3 + \frac{1}{2}(\psi_1\bar{\psi}_1 - \psi_2\bar{\psi}_2) + \frac{1}{2}(\chi_1\bar{\chi}_1 - \chi_2\bar{\chi}_2) \\
A^- &= k^- + \psi_2\bar{\psi}_1 + \chi_2\bar{\chi}_1
\end{aligned} \tag{21}$$

which belong to $(\mathbf{1}, \mathbf{0})$. It is easy to check that

$$A^i(z)A^j(w) = \frac{\delta^{ij}k'/2}{(z-w)^2} + \frac{i\epsilon^{ijk}A^k(w)}{z-w} \tag{22}$$

The second set of $su(2)$ currents can be constructed as follows

$$\begin{aligned}
B^+ &= m^+ + \chi_1\chi_2 + \psi_1\psi_2 \\
B^3 &= m^3 + \frac{1}{2}(\chi_1\bar{\chi}_1 + \chi_2\bar{\chi}_2) + \frac{1}{2}(\psi_1\bar{\psi}_1 + \psi_2\bar{\psi}_2) \\
B^- &= m^- + \bar{\chi}_2\bar{\chi}_1 + \bar{\psi}_2\bar{\psi}_1
\end{aligned} \tag{23}$$

and they satisfy

$$B^i(z)B^j(w) = \frac{\delta^{ij}k''/2}{(z-w)^2} + \frac{i\epsilon^{ijk}B^k(w)}{z-w} \tag{24}$$

which are in $(\mathbf{0}, \mathbf{1})$. The last $u(1)$ current can be constructed in a simple way

$$U(z) = -\sqrt{\frac{k'}{2}}J^0 + \sqrt{\frac{k''}{2}}K^0 = \sqrt{\frac{k' + k''}{2}}\partial Y \quad (25)$$

Here the coefficients are normalized to satisfy the large $N = 4$ algebra. The spacetime affine $su(2) \times su(2) \times u(1)$ subalgebra was given in ref. [17]

$$\{A_n^a, B_n^a, U_n\} = \oint dz \{A^a(z), B^a(z), U(z)\} \gamma^n(z) \quad (26)$$

In the light-cone gauge, they become simply

$$\{A_n^a, B_n^a, U_n\} = \oint dz \{A^a(z), B^a(z), U(z)\} e^{nq} z^{pn} \quad (27)$$

Obviously, A_n^a 's and B_n^a 's form two affine $su(2)$ Lie algebras with level $k'(k'')_{st} = pk'(pk'')$. When $q = 0, p = 1$ is set, those expression will take rather simpler form,

$$\{A_n^a, B_n^a, U_n\} = \oint dz \{A^a(z), B^a(z), U(z)\} z^n$$

So $A^a(z), B^a(z), U(z)$ can be regarded as affine Lie algebras on the spacetime's boundary. Now only four fermionic generators $Q_{\alpha\beta}$ and four supercurrents $G_{\alpha\beta}$ are to be determined. They are both in $(\frac{1}{2}, \frac{1}{2})$, so we can write them as 2×2 matrices

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}, \quad G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}, \quad (28)$$

Clearly, A^i belonging to $(\mathbf{1}, \mathbf{0})$ acts on the first index of $Q_{\alpha\beta}(G_{\alpha\beta})$ only, B^i in $(\mathbf{0}, \mathbf{1})$ acts on the second index. Q_{11} is a combination of ψ_1 and χ_1 , $Q_{11} = A\psi_1 + B\chi_1$, (we hope it will not lead to any confusions with (13)). $Q_{\alpha\beta}$ can be determined by the A^i, B^i action. The coefficients A and B can be determined by using OPE, $Q_{11}(z)Q_{22}(w) = -\frac{c}{12\gamma(1-\gamma)}\frac{1}{z-w}$. We can choose, $A = \sqrt{\frac{k'}{2}}$, $B = -\sqrt{\frac{k''}{2}}$ as the solution. So these four fermionic generators $Q_{\alpha\beta}$ can be expressed explicitly

$$\begin{aligned} Q_{11} &= \sqrt{\frac{k'}{2}}\psi_1 - \sqrt{\frac{k''}{2}}\chi_1, & Q_{12} &= \sqrt{\frac{k'}{2}}\bar{\psi}_2 - \sqrt{\frac{k''}{2}}\bar{\chi}_2 \\ Q_{21} &= \sqrt{\frac{k'}{2}}\psi_2 - \sqrt{\frac{k''}{2}}\chi_2, & Q_{22} &= -\sqrt{\frac{k'}{2}}\bar{\psi}_1 + \sqrt{\frac{k''}{2}}\bar{\chi}_1 \end{aligned} \quad (29)$$

Now, we construct the supercurrents. Consider all the possibilities for constructing G_{11} up to normalization factors,

$$G_{11} = A[k^3\psi_1 + Bk^+\psi_2 + C\partial\psi_1 + D\psi_1\bar{\psi}_2\psi_2 + EJ^0\psi_1] + F[m^3\chi_1 + Gm^+\bar{\chi}_2 + H\partial\chi_1 + I\chi_1\bar{\chi}_2\chi_2 + JK^0\chi_1] \quad (30)$$

where A, B, \dots, J (we also hope it will not lead to any ambiguities with (13)) are numeric coefficients to be determined. $A^+(z)G_{11}(w)$ regular requires $B = 1$. $B^+(z)G_{11}(w)$ regular determines $G = 1$. $G_{11}(z)G_{11}(w)$ regular determines $D = 1, E = -\sqrt{\frac{k'}{2}}$ and $I = -1, J = -\sqrt{\frac{k''}{2}}$. The coefficients A, F, C, H can be determined by considering OPE between G_{11} and A^- as well

as G_{11} and B^- . The other three supercurrents G_{21}, G_{12}, G_{22} can be obtained by acting A^- and B^- on G_{11}, \dots . In this way, we get all supercurrents and we list them as follows

$$\begin{aligned}
G_{11} &= -\sqrt{\frac{2}{k'}}[k^3\psi_1 + k^+\psi_2 + (1 - \frac{c}{6})\partial\psi_1 + \psi_1\bar{\psi}_2\psi_2 - \sqrt{\frac{k'}{2}}J^0\psi_1] \\
&\quad -\sqrt{\frac{2}{k''}}[m^3\chi_1 + m^+\bar{\chi}_2 + (1 - \frac{c}{6})\partial\chi_1 - \chi_1\bar{\chi}_2\chi_2 - \sqrt{\frac{k''}{2}}K^0\chi_1] \\
G_{21} &= -\sqrt{\frac{2}{k'}}[-k^3\psi_2 + k^-\psi_1 + (1 - \frac{c}{6})\partial\psi_2 + \bar{\psi}_1\psi_1\psi_2 - \sqrt{\frac{k'}{2}}J^0\psi_2] \\
&\quad -\sqrt{\frac{2}{k''}}[m^3\chi_2 - m^+\bar{\chi}_1 + (1 - \frac{c}{6})\partial\chi_2 - \bar{\chi}_1\chi_1\chi_2 - \sqrt{\frac{k''}{2}}K^0\chi_2] \\
G_{12} &= \sqrt{\frac{2}{k'}}[-k^3\bar{\psi}_2 + k^+\bar{\psi}_1 - (1 - \frac{c}{6})\partial\bar{\psi}_2 + \bar{\psi}_1\psi_1\bar{\psi}_2 + \sqrt{\frac{k'}{2}}J^0\bar{\psi}_2] \\
&\quad +\sqrt{\frac{2}{k''}}[m^3\bar{\chi}_2 - m^-\chi_1 - (1 - \frac{c}{6})\partial\bar{\chi}_2 - \bar{\chi}_1\chi_1\bar{\chi}_2 + \sqrt{\frac{k''}{2}}K^0\bar{\chi}_2] \\
G_{22} &= -\sqrt{\frac{2}{k'}}[k^3\bar{\psi}_1 + k^-\bar{\psi}_2 - (1 - \frac{c}{6})\partial\bar{\psi}_1 + \bar{\psi}_2\psi_2\bar{\psi}_1 + \sqrt{\frac{k'}{2}}J^0\bar{\psi}_1] \\
&\quad -\sqrt{\frac{2}{k''}}[m^3\bar{\chi}_1 + m^-\chi_2 - (1 - \frac{c}{6})\partial\bar{\chi}_1 - \bar{\chi}_2\chi_2\bar{\chi}_1 + \sqrt{\frac{k''}{2}}K^0\bar{\chi}_1]
\end{aligned} \tag{31}$$

where $c = 6k = \frac{6k'k''}{k'+k''}$ [17, 20], as desired. Generators (17), (21), (23), (25), (29), (31), form the complete basis of the large $N = 4$ SCF algebra. It is a straightforward thing to check that they satisfy the OPEs (18)

3 Middle $N = 4$ and $N = 3$ spacetime SCFT

We shall construct the so-called middle $N = 4$ and $N = 3$ superconformal algebra on the spacetime boundary in this section. For the case of middle $N = 4$ superconformal algebra, there are sixteen holomorphic currents [22, 23, 20] too, but the Kac-Moody subalgebra this time is $su(2) \times u(1)^4$ (it arises from the $S^3 \times T^4$ background).

As pointed out in ref. [20], if either of k' or k'' , say, $k'' \rightarrow \infty$ (it amounts to set $\gamma \rightarrow 1$), then in the OPE

$$B^i(z)B^j(w) = \frac{k''/2\delta^{ij}}{(z-w)^2} + \frac{i\epsilon^{ijk}B^k}{z-w}$$

we can simply neglect the last term. It means that one of the $su(2)$ Kac-Moody algebra $B^i(z)$ is broken to $u(1)^3$ Kac-Moody $\hat{B}^i(z)$. From the spacetime point of view, it corresponds to the $AdS_3 \times S^3 \times T^4$ background. The criticality requirement reduces to $k = k'$ and the center charge equals $6k$ now. We shall construct this middle $N = 4$ algebra from the large one by means of the Inönö-Wigner contraction [22].

Make the following definitions

$$\hat{T}(z) = \lim_{\gamma \rightarrow 1} T(z), \quad \hat{U}^i(z) = \lim_{\gamma \rightarrow 1} \sqrt{1 - \gamma} B^i(z)$$

$$\begin{aligned}
\hat{G}_{\alpha\beta}(z) &= \lim_{\gamma \rightarrow 1} G_{\alpha\beta}(z), & \hat{Q}_{\alpha\beta}(z) &= \lim_{\gamma \rightarrow 1} \sqrt{1-\gamma} Q_{\alpha\beta}(z) \\
\hat{J}^i(z) &= \lim_{\gamma \rightarrow 1} A^i(z), & \hat{U}(z) &= \lim_{\gamma \rightarrow 1} \sqrt{1-\gamma} U(z)
\end{aligned} \tag{32}$$

It seems that some of them are trivial and some of them have vanishing right hand sides. In fact neither of these observation are correct. The reason has been given in detail in ref.[22]. First look at $\hat{U}^i(z)\hat{U}^j(w)$, we can easily get $\hat{U}^i(z)\hat{U}^j(w) = \frac{\delta^{ij}k'/2}{(z-w)^2} = \frac{\delta^{ij}c/12}{(z-w)^2}$, so we obtain $u(1)^3$ algebra $\hat{B}^i(z)$ which is broken from $B^i(z)$ as follows,

$$\hat{B}^- = \hat{U}^-, \quad \hat{B}^3 = \hat{U}^3, \quad \hat{B}^+ = \hat{U}^+ \tag{33}$$

and other generators (energy-momentum tensor \hat{T} , supercurrent \hat{G}_{ij} and four spin $\frac{1}{2}$ fermions \hat{Q}_{ij}) can be gotten in a similar way. We list these generators in the following,

$$\begin{aligned}
\hat{T}(z) &= -\frac{1}{2}(J^0)^2 - \frac{1}{2}(K^0)^2 - \left(\frac{1}{\alpha_+} - \frac{\alpha_+}{2}\right)\partial J^0 + \frac{1}{k'} \sum k^i k^i + \frac{6}{c} \hat{U}^i \hat{U}^i + \hat{T}_F \\
\hat{G}_{11} &= -\sqrt{\frac{2}{k'}} [k^3 \psi_1 + k^+ \psi_2 + (1 - \frac{c}{6}) \partial \psi_1 + \psi_1 \bar{\psi}_2 \psi_2 - \sqrt{\frac{k'}{2}} J^0 \psi_1] \\
&\quad + (K^0 \chi_1 - \sqrt{\frac{12}{c}} \hat{U}^3 \chi_1 - \sqrt{\frac{12}{c}} \hat{U}^+ \bar{\chi}_2) \\
\hat{G}_{21} &= -\sqrt{\frac{2}{k'}} [-k^3 \psi_2 + k^- \psi_1 + (1 - \frac{c}{6}) \partial \psi_2 + \bar{\psi}_1 \psi_1 \psi_2 - \sqrt{\frac{k'}{2}} J^0 \psi_2] \\
&\quad + (K^0 \chi_2 - \sqrt{\frac{12}{c}} \hat{U}^3 \chi_2 + \sqrt{\frac{12}{c}} \hat{U}^+ \bar{\chi}_1) \\
\hat{G}_{12} &= \sqrt{\frac{2}{k'}} [-k^3 \bar{\psi}_2 + k^+ \bar{\psi}_1 - (1 - \frac{c}{6}) \partial \bar{\psi}_2 + \bar{\psi}_1 \psi_1 \bar{\psi}_2 + \sqrt{\frac{k'}{2}} J^0 \bar{\psi}_2] \\
&\quad + (K^0 \bar{\chi}_2 + \sqrt{\frac{12}{c}} \hat{U}^3 \bar{\chi}_2 - \sqrt{\frac{12}{c}} \hat{U}^- \chi_1) \\
\hat{G}_{22} &= -\sqrt{\frac{2}{k'}} [k^3 \bar{\psi}_1 + k^- \bar{\psi}_2 - (1 - \frac{c}{6}) \partial \bar{\psi}_1 + \bar{\psi}_2 \psi_2 \bar{\psi}_1 + \sqrt{\frac{k'}{2}} J^0 \bar{\psi}_1] \\
&\quad - (K^0 \bar{\chi}_1 + \sqrt{\frac{12}{c}} \hat{U}^3 \bar{\chi}_1 + \sqrt{\frac{12}{c}} \hat{U}^- \chi_2) \\
\hat{U}(z) &= \sqrt{\frac{c}{12}} K^0 \\
\hat{B}^- &= \hat{U}^-, \quad \hat{B}^3 = \hat{U}^3, \quad \hat{B}^+ = \hat{U}^+ \\
\hat{Q}_{11} &= -\sqrt{\frac{c}{12}} \chi_1, \quad \hat{Q}_{12} = -\sqrt{\frac{c}{12}} \bar{\chi}_2 \\
\hat{Q}_{21} &= -\sqrt{\frac{c}{12}} \chi_2, \quad \hat{Q}_{22} = \sqrt{\frac{c}{12}} \bar{\chi}_1 \\
\hat{J}^+ &= k^+ + \psi_1 \bar{\psi}_2 + \chi_1 \bar{\chi}_2 \\
\hat{J}^3 &= k^3 + \frac{1}{2}(\psi_1 \bar{\psi}_1 - \psi_2 \bar{\psi}_2) + \frac{1}{2}(\chi_1 \bar{\chi}_1 - \chi_2 \bar{\chi}_2) \\
\hat{J}^- &= k^- + \psi_2 \bar{\psi}_1 + \chi_2 \bar{\chi}_1
\end{aligned} \tag{34}$$

where

$$\hat{T}_F = T_F = \frac{1}{2}(\partial\psi_1\bar{\psi}_1 - \psi_1\partial\bar{\psi}_1 + \partial\psi_2\bar{\psi}_2 - \psi_2\partial\bar{\psi}_2 + \partial\chi_1\bar{\chi}_1 - \chi_1\partial\bar{\chi}_1 + \partial\chi_2\bar{\chi}_2 - \chi_2\partial\bar{\chi}_2)$$

are unchanged.

They satisfy the following middle $N = 4$ superconformal algebra

$$\begin{aligned} \hat{T}(z)\hat{T}(w) &= \frac{c/2}{(z-w)^4} + \frac{2\hat{T}(w)}{(z-w)^2} + \frac{\partial\hat{T}(w)}{z-w} \\ \hat{G}_{\alpha\beta}(z)\hat{G}_{\alpha'\beta'}(w) &= \epsilon_{\alpha\alpha'}\epsilon_{\beta\beta'}\left[\frac{2c/3}{(z-w)^3} + \frac{2\hat{T}(w)}{z-w}\right] + (\sigma_i)_\alpha{}^\rho\epsilon_{\rho\alpha'}\epsilon_{\beta\beta'}\left[\frac{4\hat{J}^i}{(z-w)^2} + \frac{2\partial\hat{J}^i}{z-w}\right](w) \\ \hat{J}^i(z)\hat{G}_{\alpha\beta}(w) &= \frac{1}{2}(\sigma_i)_\alpha{}^\rho\frac{\hat{G}_{\rho\beta}(w)}{z-w} \\ \hat{B}^i(z)\hat{G}_{\alpha\beta}(w) &= \frac{(\sigma_i)_\beta{}^\rho\hat{Q}_{\alpha\rho}(w)}{(z-w)^2} \\ \hat{U}(z)\hat{G}_{\alpha\beta} &= \frac{\hat{Q}_{\alpha\beta}(w)}{(z-w)^2} \\ \hat{J}^i(z)\hat{J}^j(w) &= \frac{k'/2\delta^{ij}}{(z-w)^2} + \frac{i\epsilon^{ijk}\hat{J}^k(w)}{z-w} \\ \hat{B}^i(z)\hat{B}^j(w) &= \frac{c\delta^{ij}}{12(z-w)^2} \\ \hat{Q}_{\alpha\beta}(z)\hat{G}_{\alpha'\beta'}(w) &= \frac{\epsilon_{\alpha\alpha'}\epsilon_{\beta\beta'}\hat{U}(w)}{z-w} - \frac{(\sigma_i)_{\beta'}{}^\rho\epsilon_{\rho\beta}\epsilon_{\alpha\alpha'}\hat{B}^i(w)}{z-w} \\ \hat{Q}_{\alpha\beta}(z)\hat{Q}_{\alpha'\beta'}(w) &= -\frac{c/12\epsilon_{\alpha\alpha'}\epsilon_{\beta\beta'}}{z-w} \\ \hat{J}^i(z)\hat{Q}_{\alpha\beta}(w) &= \frac{1}{2}(\bar{\sigma}_i)_\alpha{}^\rho\frac{\hat{Q}_{\rho\beta}(w)}{z-w} \\ \hat{U}(z)\hat{U}(w) &= -\frac{c/12}{(z-w)^2} \\ \hat{U}^i(z)\hat{U}^j(w) &= \frac{c/12\delta^{ij}}{(z-w)^2} \\ \hat{T}(z)\hat{\Phi}(w) &= \frac{d_{\hat{\Phi}}\hat{\Phi}(w)}{(z-w)^2} + \frac{\partial\hat{\Phi}(w)}{z-w} \end{aligned} \tag{35}$$

where $\hat{\Phi} = \{\hat{G}_{\alpha\beta}, \hat{B}^i, \hat{U}, \hat{J}^i, \hat{Q}_{\alpha\beta}\}$, accordingly, $d_{\hat{\Phi}}$ equal to $\{\frac{3}{2}, 1, 1, 1, \frac{1}{2}\}$, and $c = 6k$, as mentioned above.

Recently, string theory on the $AdS_3 \times (S^3 \times S^3 \times S^1)/Z_2$ was investigated in ref.[24], and the spacetime $N = 3$ superconformal theories was constructed. We shall construct the same algebra in terms of the transversal physical degrees of freedom.

We set $k' = k''$ in our large $N = 4$ algebra. Once this setting is performed, there is an automorphism in our large $N = 4$ algebra, namely, a Z_2 action[24],

$$(A^i(z), B^i(z), Y(z)) \rightarrow (B^i(z), A^i(z), -Y(z))$$

We consider the eigenstates of the Z_2 action. Those with eigenvalue $+1$ are invariant under Z_2 and will form a subalgebra of the large $N = 4$ algebra. It can be checked that the diagonal part of the $su(2) \times su(2)$ algebra is invariant under Z_2 and the isometry $su(2) \times su(2)$ of the large $N = 4$ is reduced to the diagonal $su(2)$.

The diagonal $su(2)$ is invariant under Z_2 , so we can construct our affine Lie algebra uniquely by summing those two $su(2)$ s

$$J^i(z) = A^i(z) + B^i(z)$$

Supercharges G_{ij} belong to the $(\frac{1}{2}, \frac{1}{2})$ representation of $su(2) \times su(2)$, and they are decomposed into $\mathbf{1} \oplus \mathbf{0}$ under the diagonal $su(2)$. Now we shall explain that only the triplet is Z_2 invariant. Under the Z_2 action, two $su(2)$ s are interchanged and at the same time Y is reflected. In our free field realization, this corresponds to A^i and B^i interchanged, and U reflected simultaneously according to (25). Supercurrents and four fermions transform as

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \rightarrow - \begin{pmatrix} Q_{11} & Q_{21} \\ Q_{12} & Q_{22} \end{pmatrix}, G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \rightarrow \begin{pmatrix} G_{11} & G_{21} \\ G_{12} & G_{22} \end{pmatrix} \quad (36)$$

It is not very difficult to check that after Z_2 action, those set of generators still satisfy OPEs (18), except A^i, B^i interchanged, and the index of supercurrents also exchanged. So only triplet is invariant under the Z_2 action. Closure of this algebra needs a spin $\frac{1}{2}$ fermionic current Ψ . We list our $N = 3$ algebra generators in the following

$$\begin{aligned} T(z) &= -\frac{1}{2}(J^0)^2 - \frac{1}{2}(K^0)^2 - \frac{\sqrt{2}}{2}\left(\frac{1}{\alpha_+} - \frac{\alpha_+}{2}\right)\partial J^0 - \frac{\sqrt{2}}{2}\left(\frac{1}{\alpha_+} - \frac{\alpha_+}{2}\right)\partial K^0 \\ &\quad + \frac{1}{k'} \sum k^i k^i + \frac{1}{k''} \sum m^i m^i + T_F \\ T_F &= \frac{1}{2}(\partial\psi_1\bar{\psi}_1 - \psi_1\partial\bar{\psi}_1 + \partial\psi_2\bar{\psi}_2 - \psi_2\partial\bar{\psi}_2 + \partial\chi_1\bar{\chi}_1 - \chi_1\partial\bar{\chi}_1 + \partial\chi_2\bar{\chi}_2 - \chi_2\partial\bar{\chi}_2) \\ J^+ &= A^+ + B^+, \quad J^3 = A^3 + B^3, \quad J^- = A^- + B^- \\ G^+ &= i\sqrt{2}G_{11}, \quad G^3 = -i\frac{\sqrt{2}}{2}(G_{12} + G_{21}), \quad G^- = -i\sqrt{2}G_{22} \\ \Psi &= -i\frac{\sqrt{2}}{2}(Q_{21} - Q_{12}) \end{aligned} \quad (37)$$

where the coefficients are normalized to satisfy the following OPEs

$$\begin{aligned} T(z)T(w) &= \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \\ J^i(z)J^j(w) &= \frac{k'\delta^{ij}}{(z-w)^2} + \frac{i\epsilon^{ijk}J^k(w)}{z-w} \\ J^i(z)G^j(w) &= \frac{i\epsilon^{ijk}G^k(w)}{z-w} + \frac{\delta^{ij}\Psi(w)}{(z-w)^2} \\ G^i(z)G^j(w) &= \frac{2c/3\delta^{ij}}{(z-w)^3} + \frac{2i\epsilon^{ijk}J^k(w)}{(z-w)^2} + \frac{i\epsilon^{ijk}\partial J^k}{z-w} + \frac{2\delta^{ij}T(w)}{z-w} \\ \Psi(z)\Psi(w) &= \frac{k'}{z-w} \\ T(z)\hat{\Phi}(w) &= \frac{d_{\hat{\Phi}}\hat{\Phi}(w)}{(z-w)^2} + \frac{\partial\hat{\Phi}(w)}{z-w} \end{aligned} \quad (38)$$

where $\hat{\Phi} = \{J^i, G^i, \Psi\}$, $d_{\hat{\Phi}} = \{1, \frac{3}{2}, \frac{1}{2}\}$, and the center charge now is $c = 6k = 3k' = 3k''$ (remember $k = \frac{k'}{2} = \frac{k''}{2}$ in this case).

4 Conclusions and Remarks

To have a better understanding of the *AdS/CFT* correspondence, it is essential to construct the spacetime boundary CFT explicitly. In this note we have realized the spacetime SCF algebras explicitly in terms of transversal degrees of freedom, corresponding to strings propagating on $AdS_3 \times S^3 \times S^3 \times S^1$, $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times (S^3 \times S^3 \times S^1)/Z_2$ respectively. This is a modest step towards to full construction of the spacetime CFT living on the AdS_3 boundary. As in the case of the large $N = 4$ SCF algebra [20], we may consider the spacetime SCFT as some twisted version of p copies of nonlinear sigma model, each with center charge $6k$. It remains to check the correspondence between the string theory in the bulk and the CFT on the boundary.

A manifest similarity between the two theory is that the DDF state [25, 26] on the string theory side resembles very much to the conformal algebra on the boundary CFT side, except that the energy momentum tensor on the each side differ by a improved term. Does that means some duality between the two CFT? It deserves further clarification.

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